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Tolman Energy of a Stringy Charged Black Hole

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Virbhadra and Parikh studied the energy distribution associated with a stringy charged black hole in Einstein's prescription. We study this using Tolman's energy-momentum complex and get the same result as obtained by Virbhadra and Parikh. The entire energy is confined inside the black hole.

Energy-momentum localization has been one of the most important and challenging research topics in the general theory of relativity (Møller, 1958; Hawking, 1968; Penrose, 1982; Rosen, 1994; and references therein). Following the Einstein energy-momentum complex, many definitions of energy, momentum, and angular momentum for a general relativistic system have been proposed (Brown and York, 1993; Aguirregabiria *et al.*, 1996). The physical interpretation of these nontensorial energy-momentum complexes has been questioned by a number of physicists, including Weyl, Pauli, and Eddington (see Chandrasekhar and Ferrari, 1991). There is a suspicion that different energy-momentum complexes could give different energy distributions in a given spacetime. To this end Virbhadra and his collaborators have considered many spacetimes and have shown that several energy-momentum complexes give the same acceptable result for a given spacetime. Virbhadra (1990a–c) considered the Kerr–Newman metric and carried out calculations up to the third order of the rotation parameter. Cooperstock and Richardson (1991) extended his investigations up to the seventh order of the rotation parameter and reported that several definitions yield the same result. Virbhadra (1992) carried out calculations for the Vaidya (radiating Schwarzschild) metric and found that several definitions give the same result. Virbhadra and Parikh (1994), after obtaining a conformal scalar dyon black hole solution, calculated

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the energy distribution with this black hole and found a reasonable result. Chamorro and Virbhadra (1995) studied the Bonnor–Vaidya spacetime and found that energy-momentum complexes give the same result as obtained by Tod using the Penrose definition (for details see Aguirregabiria *et al.*, 1996). Rosen and Virbhadra (1993) and Virbhadra (1995) investigated the well-known Einstein–Rosen spacetime and found that several energy-momentum complexes give the same result for the energy and energy current densities. Aguirregabiria *et al.* (1996) showed that several well-known energy-momentum complexes coincide for any Kerr–Schild class metric. These are definitely encouraging results.

For the Reissner–Nordstrom metric, several definitions of energy give

$$E(r) = M - \frac{Q^2}{2r} \tag{1}$$

(Tod, 1983; Hayward, 1994; Aguirregabiria *et al.*, 1996). Thus the energy is both in its interior and exterior. There is considerable difference between the solutions in the Einstein–Maxwell theory and the low-energy string theory. Virbhadra and Parikh (1993) deduced, using Einstein's prescription, that the gravitational energy of a stringy charged black hole is given by $E = M$, and thus the energy is confined to the interior of the black hole. It is worth investigating whether or not other definitions of energy give the same result as obtained by them. Throughout this paper we use $G = 1$ and $c = 1$ units and follow the convention that Latin indices take values from 0 to 3 and Greek indices take values from 1 to 3.

For a static, spherically symmetric charged black hole in low-energy string theory, the line element is given by (Garfinkle *et al.*, 1991)

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - \left(1 - \frac{\alpha}{r}\right) r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \tag{2}$$

where

$$\alpha = Q^2 \frac{\exp(-2\phi_0)}{M} \tag{3}$$

M and Q are, respectively, mass and charge parameters; ϕ_0 is the asymptotic value of the dilaton field. It is well known that the energy-momentum complexes give meaningful result if calculations are performed in quasi-Cartesian coordinates. The line element (2) may be transformed to quasi-Cartesian coordinates:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{\alpha}{r}\right)(dx^2 + dy^2 + dz^2)$$
$$- \frac{(1 - 2M/r)^{-1} - (1 - \alpha/r)}{r^2} (x\,dx + y\,dy + z\,dz)^2 \tag{4}$$

according to

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right), \qquad \phi = \tan^{-1}(y/x) \tag{5}$$

The energy-momentum complex of Tolman (1934) is

$$\mathcal{T}_k^i = \frac{1}{8\pi} U_{k,j}^{ij} \tag{6}$$

where

$$U_k^{ij} = \sqrt{-g}\left[-g^{pi}\left(-\Gamma_{kp}^j + \frac{1}{2} g_k^j \Gamma_{ap}^a + \frac{1}{2} g_p^j \Gamma_{ak}^a\right)\right.$$
$$\left. + \frac{1}{2} g_k^j g^{pm}\left(-\Gamma_{pm}^j + \frac{1}{2} g_p^j \Gamma_{am}^a + \frac{1}{2} g_m^j \Gamma_{ap}^a\right)\right] \tag{7}$$

$\mathcal{T}_0^0$ is the energy density, $\mathcal{T}_0^\alpha$ are the components of energy current density, and $\mathcal{T}_\alpha^0$ are the momentum density components. The energy E is given by

$$E = \iiint \mathcal{T}_0^0\, dx\, dy\, dz \tag{8}$$

Using the Gauss theorem (noting that the spacetime under consideration is static), one has

$$E = \frac{1}{8\pi} \iint U_0^{0\beta} n_\beta\, dS \tag{9}$$

n_β stands for the 3-components of the unit vector over an infinitesimal surface element dS. For the metric given by equation (4), Virbhadra and Parikh (1993) computed the determinant of the metric tensor and the contravariant components of the tensor. To compute the energy using (9), we require the following list of nonvanishing components of the Christoffel symbol:

$$
\begin{aligned}
\Gamma^0_{01} &= \frac{Mx}{r^2(r-2M)}, & \Gamma^0_{02} &= \frac{My}{r^2(r-2M)} \\
\Gamma^0_{03} &= \frac{Mz}{r^2(r-2M)}, & \Gamma^1_{00} &= \frac{M(r-2M)x}{r^4} \\
\Gamma^2_{00} &= \frac{M(r-2M)y}{r^4}, & \Gamma^3_{00} &= \frac{M(r-2M)z}{r^4} \\
\Gamma^1_{11} &= x(a_1 + a_2x^2), & \Gamma^2_{11} &= y(a_3 + a_2x^2) \\
\Gamma^3_{11} &= z(a_3 + a_2x^2), & \Gamma^1_{22} &= x(a_3 + a_2y^2) \\
\Gamma^2_{22} &= y(a_1 + a_2y^2), & \Gamma^3_{22} &= z(a_3 + a_2y^2) \\
\Gamma^1_{33} &= x(a_3 + a_2z^2), & \Gamma^2_{33} &= y(a_3 + a_2z^2) \\
\Gamma^3_{33} &= z(a_1 + a_2z^2), & \Gamma^1_{12} &= y(a_4 + a_2x^2) \\
\Gamma^1_{13} &= z(a_4 + a_2x^2), & \Gamma^2_{21} &= x(a_4 + a_2y^2) \\
\Gamma^2_{23} &= z(a_4 + a_2y^2), & \Gamma^3_{31} &= x(a_4 + a_2z^2) \\
\Gamma^3_{32} &= y(a_4 + a_2z^2) \\
\Gamma^1_{23} &= \Gamma^2_{13} = \Gamma^3_{12} = a_1xyz
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
a_1 &= \frac{1}{2r^4}\left(\alpha r + 4Mr - 2\alpha M + \frac{2\alpha r^2}{r-\alpha}\right) \\
a_2 &= \frac{1}{2r^6}\left(2\alpha M - 3\alpha r - 4Mr - \frac{2Mr}{r-2M} - \frac{2\alpha^2 r}{r-\alpha}\right) \\
a_3 &= \frac{1}{2r^4}(\alpha r + 4Mr - 2\alpha M) \\
a_4 &= \frac{\alpha}{2r^2(r-\alpha)}
\end{aligned} \tag{11}
$$

Further, after straightforward, but very lengthy calculations, we get

$$
\begin{aligned}
U_0^{01} &= 2Mx/r^3 \\
U_0^{02} &= 2My/r^3 \\
U_0^{03} &= 2Mz/r^3
\end{aligned} \tag{12}
$$

Now using (12), $n_\beta = (x/r, y/r, z/r)$, and $dS = r^2 \sin\theta d\theta\, d\phi$ in (9), we get

$$E = M \tag{13}$$

Thus the entire energy of a charged black hole in low-energy string theory lies inside the black hole. The "effective gravitational mass," given by equation (13), that a neutral test particle experiences is always positive. This is quite different from the case of the Reissner–Nordstrom metric [for $r < Q^2/2M$ the "effective gravitational mass" $E(r)$ is negative]. Further, it is worth noting that, using the Tolman energy-momentum complex, we got the same result as obtained by Virbhadra and Parikh (they used the Einstein energy-momentum complex). Our result supports the importance of energy-momentum complexes.

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